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ABSTRACT

Details of arithmetic topics proposed for inclusion in a modern elementary mathematics program and a rationale for the selection of these topics are given. The sequencing of the topics is discussed. (Author/DT)

Working Paper No. 79

# The Content of Arithmetic Included in a Modern Elementary Mathematics Program



Report from the Project on Individually Guided Elementary Mathematics, Phase 2: Analysis Of Mathematics Instruction

**Wisconsin Research and Development**
**CENTER FOR COGNITIVE LEARNING**

**THE UNIVERSITY OF WISCONSIN**
**Madison, Wisconsin**

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Working Paper No. 79

# THE CONTENT OF ARITHMETIC INCLUDED IN A MODERN ELEMENTARY MATHEMATICS PROGRAM

by

John G. Harvey

Report from the Project on

Individually Guided Elementary Mathematics

Phase Two, Analysis of Mathematics Instruction

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October 1971

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This working paper is from the Project on Individually Guided Elementary Mathematics in Program 2. General objectives of the program are to establish rationale and strategy for developing instructional systems, to identify sequences of concepts and cognitive skills, to develop assessment procedures for those concepts and skills, to identify or develop instructional materials associated with the concepts and cognitive skills, and to generate new knowledge about instructional procedures. Contributing to the program objectives, the mathematics project has developed and tested a televised course in arithmetic for grades 1-6 which provides not only a complete program of instruction for the pupils but also inservice training for teachers. Analysis of Mathematics Instruction is currently the only active phase of the mathematics project and has a long-term goal of providing an individually guided instructional program in elementary mathematics. Preliminary activities include identifying instructional objectives, student activities, teacher activities materials, and assessment procedures for integration into a total mathematics curriculum. The third phase focused on the development of a computer system for managing individually guided instruction in mathematics and on a later extension of the system's applicability.

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# ABSTRACT

This paper details the arithmetic topics proposed for inclusion in a modern elementary mathematics program, gives a rationale for the selection of these topics, and discusses the sequencing of the topics.

# I

# INTRODUCTION

The purpose of the Analysis of Mathematics Instruction Project is to generate knowledge about mathematics instruction and to incorporate it into a research-based individually guided instructional program for elementary mathematics. Intermediate objectives of this project are: (1) to establish a decision-making procedure that takes into account both learner and subject matter, and which will identify the topics from mathematics which are to be incorporated into an elementary level instructional program; (2) to develop a sequential outline of concepts and cognitive skills for the topics; (3) to explain the fundamental learning processes associated with concepts and skills; (4) to identify materials, teacher activities, and pupil activities and to locate or develop the instructional aids, teacher guides, student materials, and testing program necessary to implement the instructional program; (5) to develop assessment instruments and criteria for each concept and skill; and (6) to integrate the above materials and procedures into a prototypic mathematics curriculum, Developing Mathematical Processes. (For a comprehensive overview of the project see Romberg and Harvey, 1969, and Harvey, Romberg, and Fletcher, 1969.)

When complete, Developing Mathematical Processes (DMP) will include material from three subject matter areas: arithmetic, geometry, and probability and statistics. DMP will be designed to foster logical validation, to promote problem solving abilities, and to identify and encourage mathematical creativity. It is the purpose of the current paper to detail the



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arithmetic topics proposed for inclusion in a modern elementary mathematics program, a rationale for the selection of these topics, and a discussion of the sequencing of the topics.

## II

## ARITHMETIC IN THE ELEMENTARY SCHOOL

For several centuries the study of arithmetic has been a part of the curriculum of every man or child who wished to acquire a formal education. As a matter of fact on some occasions the ability of a man to cipher correctly has been a portion of the measure used to determine whether he was an educated man. Since the early Colonial period in the United States it has been true that arithmetic, reading, and writing are the primary focal points of the early elementary years and usually occupy a prominent position in the curriculum for grades 4-6.

Historically, all concerned parties seem to have agreed that because of its social utility, its contribution to mental discipline, and its usefulness in all branches of mathematics, arithmetic should be taught to the child at an early age and that his knowledge and skills should be highly developed by the time he enters the secondary school. Addressing the American Institute of Instruction in August, 1830, Colburn said,

> The subject [arithmetic] is certainly an important one in every point of view, whether we consider its application to the affairs of life, or its effect as a discipline of the mind, or the time which is usually devoted to it.
>
> With regard to its application, there are very few persons, either male or female, arrived at years of discretion, who have not occasions daily to make use of arithmetic in some form

> or other in the ordinary routine of business. And the person the most ready in calculation is much the most likely to succeed in business of any kind. As our country becomes more thickly peopled, and competition in the various branches of business becomes greater, and further progress is made in the arts, and new arts are discovered, knowledge of all kinds is brought into requisition; and none more so than that of arithmetic, and the higher branches of mathematics, of which arithmetic is the foundation.
>
> Arithmetic, when properly taught, is acknowledged by all to be very important as a discipline of the mind; so much so that even if it had no practical application which should render it valuable on its own account, it would still be well worth while to bestow a considerable portion of time on it for this purpose alone. This is a very important consideration, though a secondary one compared with its practical utility. (Bidwell and Clason, 1970, p. 24)

Two reports of the National Education Association, the 1893 Report of the Committee on Secondary School Studies and the 1895 Report of the Committee of Fifteen on Elementary Education, reaffirmed the pertinence of these reasons for teaching arithmetic (Bidwell and Clason, 1970). The 1923 report of the Mathematical Association of America, The Reorganization of Mathematics in Secondary Education, strongly emphasized the practical and cultural aims of mathematics education although it did de-emphasize the disciplinary aims (Bidwell and Clason, 1970). The report of the Committee on the Function of Mathematics in General Education in 1938 concurred with this view as did a Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics in 1940, the Commission on Post-War Plans in 1944 (Bidwell and Clason, 1970), and the Cambridge Conference on School Mathematics in 1963 (Educational Services Incorporated, 1963).

These authoritative opinions and the bulk of attention to arithmetic contained therein lead to the conclusion that arithmetic should be the

core of the elementary mathematics program and in particular of Developing Mathematical Processes. There is little, if any, need to further justify the inclusion of arithmetic topics in an elementary school mathematics curriculum. Instead the questions should be those of what, when and how. Specifically, what arithmetic is to be studied? In what sequence are the selected topics to be arranged? And what pedagogy is to be used? The remainder of this paper will delve into the first two of these questions.

## III

## THE ARITHMETIC CONTENT OF A MODERN ELEMENTARY MATHEMATICS PROGRAM

In a previous paper (Harvey, Meyer, Romberg, and Fletcher, 1969) three criteria were given for selecting topics for inclusion in an elementary mathematics program. As stated in that paper, each topic must:

1. Contribute to and integrate with the other portions of the mathematics program;
2. Involve the kinds of behaviors which an elementary school child could be expected to exhibit; and
3. Be useful at the higher grades in developing mathematical intuition, strategies, knowledge, and maturity.

In this section of the paper the arithmetic topics which might be included in a modern elementary mathematics program will be described and compared to the stated criteria. Finally, the sequencing of the topics will be discussed.

Arithmetic is a generic word in mathematics which does not describe any specific body of content. Loosely speaking, the word arithmetic refers to the algebraic structure which results when a set of objects has defined on it certain operations and relations. This paper confines its attention to the sets of natural numbers, counting numbers, integers, rational numbers, real numbers and complex numbers; to the operations of addition and multiplication; and to the relations of equality and order.

Further, to assure a common understanding of terms, the study of arithmetic of one of these sets and its associated operations and relations is defined to be the study of those numbers as objects and as symbolic representations of real-world situations, the study of the properties of the operations and relations as defined on that set and the interactions of the operations and relations with each other, and the means of performing the operations and of comparing with respect to the relations.

With these boundaries in mind the question, "What arithmetic is to be taught?" becomes "Shall the arithmetic studied in a modern elementary mathematics program be that of the natural numbers, the counting numbers, the integers, the rational numbers, the real numbers, or the complex numbers?"

However, the question is not as complex as might be anticipated by the layman since these sets are ordered by set inclusion (see figure 1) and since the operations and relations defined on one set are extensions of the operations and relations defined on each of its subsets. So the question can be further limited to: "By the time a child completes a modern elementary mathematics program of what set and its subsets shall he have complete arithmetic knowledge? And for the larger sets (if any) what shall be his knowledge of them and of their operations and relations?"

Criterion 1 (page 6) is of no assistance in light of the decision to make the study of arithmetic the core of the curriculum being proposed except that since geometry and probability and statistics are to be included, the arithmetic study must not be so ambitious as to eliminate these subject matter areas.



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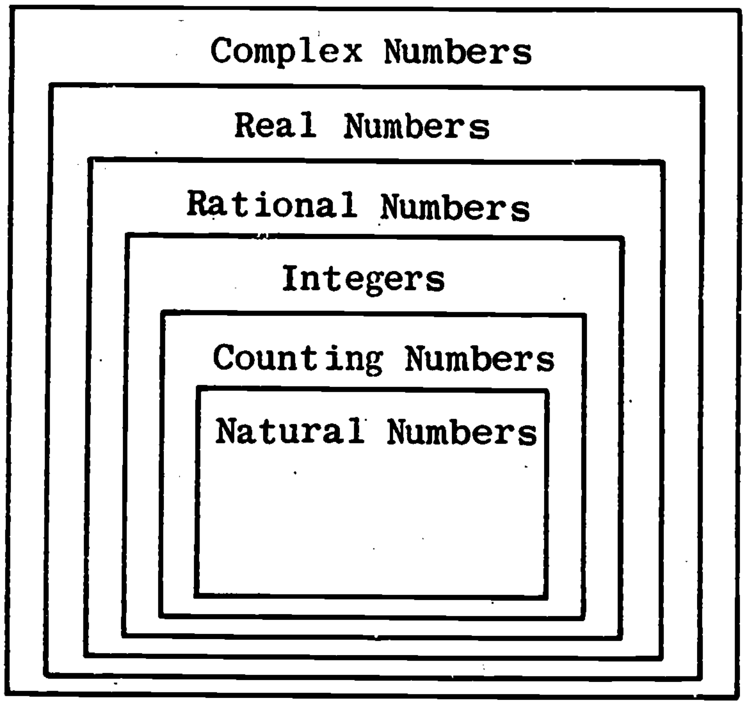


Sets of Numbers Ordered by Set Inclusion

Figure 1

Criterion 2 is more useful. As far as the author can tell, there is no research which successfully demonstrates that all of the behaviors involved in learning arithmetic can be exhibited by elementary school children. However, there is a considerable body of research on arithmetic, especially the arithmetic of the counting numbers and of the non-negative common fractions and decimal fractions (a subset of the rational numbers) which indicates that children can be taught this subject matter successfully. (For review of the research on arithmetic the reader should see Suppes, Jerman, and Brian, 1968, and Suydam and Weaver, 1970). And since the arithmetics of the counting numbers and the non-negative rational numbers have been taught for so long to elementary school students, it can be inferred

that children can exhibit acceptable behaviors. There is little evidence that negative numbers can be learned except the informal evidence obtained by elementary schools over the past ten years as they have attempted to implement new mathematics programs containing this topic. There is even less evidence that the arithmetic of the real numbers and the complex numbers can be taught successfully. It would seem that the development of the real numbers in the elementary school may not be possible or practical because of the difficulties of defining them for a child so that simultaneously the intuitive definition given retains mathematical integrity and the child can understand and use it. However, it is probably possible to introduce some pre-mathematical ideas about the real numbers. Thus it would seem that it may be possible to satisfy criterion 2 by studying the arithmetic of the rational numbers and by providing at least an intuitive knowledge of the set of real numbers or at least of particular real numbers including some of the algebraic numbers and the transcendental numbers.

Unless this latter can be done it is impossible to introduce significant metric geometry associated with areas and volumes into the elementary mathematics curriculum. However, there seems to be little need to rigorously define the real numbers or to explore algorithms for adding them or multiplying them at this stage even though the completeness of the set of real numbers is used very heavily beginning in the junior high school. Instead the concept of completeness can be acquired intuitively using the Cartesian correspondence of rational numbers to points on lines and the continuity of lines.

Finally then, applying criterion 3, the arithmetic studied should be as rich in properties as possible because the development of most of the other

branches of mathematics depends on sets with arithmetics that have a generous supply of properties.

All of the arithmetics being considered in this paper have the following properites:

1. The addition is closed, associative and commutative.
2. The multiplication is closed, associative and commutative.
3. The multiplication is distributive over the addition.
4. The equality relation is reflexive, symmetric and transitive and is unchanged by addition and multiplication. In all but the complex number system, every pair of numbers can be compared and ordered and this ordering is rich in properties.

The properties on which the arithmetics are different are summarized in table 1; in that table N denotes the natural numbers, C the counting numbers, Z the integers, Q the rational numbers, R the real numbers, and R(i) the complex numbers. From this summary it can be seen that the rational numbers and the real numbers have similar algebraic properties--each has an additive identity and all of the appropriate elements of each have additive inverses and multiplicative inverses. Also it can be noted that the natural numbers, the counting numbers, and the integers do not have these specified properties and that the rational numbers have all of the order properties of the real numbers except completeness.

Keeping in mind the difficulty of defining the real numbers and the comparative richness of the rational numbers, it is concluded that the primary arithmetic goal of a modern elementary mathematics program should be the study of the arithmetic of the rational numbers and that a worthwhile

secondary goal would be to develop an intuitive knowledge of the set of real numbers and its arithmetic.

Table 1

SUMMARY OF PROPERTIES ON WHICH THE ARITHMETICS ARE DIFFERENT

| Properties | N | C | Z | Q | R | R(i) |
|---|---|---|---|---|---|---|
| Additive identity | No | Yes | Yes | Yes | Yes | Yes |
| Additive inverses | No | No | Yes | Yes | Yes | Yes |
| Multiplicative inverses | No | No | No | Yes | Yes | Yes |
| Dense | No | No | No | Yes | Yes | Yes |
| Complete | No | No | No | No | Yes | Yes |
| Algebraically closed | No | No | No | No | No | Yes |

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IV

# INSTRUCTIONAL SEQUENCES FOR THE ARITHMETIC CONTENT

The decision to study the arithmetic of the rational numbers and to develop an intuitive knowledge of the real numbers does not complete the task of specifying the arithmetic component of a modern elementary mathematics program. In order to complete this task the mathematics must be divided into units and these units must be sequenced.

## The Traditional Sequence

Traditionally, the study began with counting and numeration. Then the operations defined on the natural numbers were studied in the following order: addition, subtraction, multiplication, and division. This study has normally constituted the material covered in the primary grades. The topics covered in the upper elementary grades have been common fractions, decimal fractions, the integers, and the operations defined on each of these sets. No definitive order is established for these later topics. Table 2 gives the approximate sequencing of selected topics as included in The American Calculator (Slocomb, 1831), University Arithmetic (Davies, 1870), The Report of the Committee of Fifteen on Elementary Education (National Education Association, 1895), The Public School Arithmetic for Grammar Grades (McLellan and Ames, 1902), the report of the Committee of Seven on Grade-Placement in Arithmetic (Washburne, 1939), Patterns in Arithmetic (Van Engen, 1967), Mathematics for the Elementary School (School Mathematics Study Group, 1965), Elementary School Mathematics

Table 2

REPRESENTATIVE SEQUENCING OF ARITHMETIC TOPICS

| | Sources | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Topics | Slocomb | Davies | Committee of 15 | McLellan and Ames | Committee of 7 | PIA | SMSG | ESM | GCMP |
| Sets | * | * | * | * | * | 1 | 1 | 1 | 1 |
| Numeration | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| Place Value | 2 | 2 | 2 | 2 | * | 5 | 6 | 6 | 6 |
| Counting Numbers Order | * | * | * | 7 | * | 3 | 3 | 3 | 3 |
| Addition | 3 | 3 | 3 | 3 | 2 | 4 | 4 | 4 | 4 |
| Subtraction | 4 | 4 | 4 | 4 | 3 | 6 | 5 | 5 | 5 |
| Multiplication | 5 | 5 | 5 | 5 | 4 | 7 | 7 | 7 | 7 |
| Division | 6 | 6 | 7 | 6 | 5 | 8 | 8 | 8 | 8 |
| Common Fractions | 8 | 7 | 6 | 8 | 7 | 9 | 9 | 9 | 9 |
| Decimal Fractions | 7 | 8 | 8 | 9 | 6 | 10 | 10 | 10 | 10 |
| Integers | * | * | * | * | * | 11 | 11 | 11 | 11 |

* Does not appear

(Eicholz and O'Daffer, 1964), and the Greater Cleveland Mathematics Program (Educational Research Council of Greater Cleveland, 1964).

The listing in table 2 is not completely accurate and does not tell a complete story. In each case the sequence number was assigned to a topic when it seemed that the topic was first extensively treated and initial mastery was expected. Thus, the table does not necessarily indicate the first time a particular topic is informally or intuitively introduced nor does it detail the reappearance of a topic after mastery was initially expected. At times it is almost impossible to detect informal or intuitive introductions. Also, it is assumed that every mathematics program provides periodic restudy to assist in retaining previously mastered knowledge. Furthermore, the point at which initial mastery of a concept is expected is most important. Once this point is determined, then the amount and kind of informal premastery learning can be described and the necessary retention study determined; that is, while the specification of the point at which mastery learning is expected does not uniquely describe all activities related to a given topic, this is the most important point and its choice strongly influences all other activities.

When table 2 is examined it can be seen that there has been little variation in topical arrangement since 1831. The sequence described by Patterns in Arithmetic (Van Engen, 1967), by the texts produced by the School Mathematics Study Group (1965), in Elementary School Mathematics (Eicholz and O'Daffer, 1964), and for the books of the Educational Research Council of Greater Cleveland (1964) differs very little from that given in The American Calculator (Slocomb, 1831) except to begin by studying sets, to introduce the idea of number base and place value toward the

middle of the program, and to include topics dealing with the negative integers. While it may be true that this is the only viable instructional sequence, there are several others which may be possible; certainly the mathematical content decided upon does not describe a unique sequencing. However, there are two important factors which must be considered for they considerably reduce the possible sequences. The two factors are the goals selected for mathematics instruction and the implications to be obtained from developmental psychology.

## Goals for Mathematics Instruction and Implications of Developmental Psychology

As was previously mentioned the goals for mathematics instruction have quite frequently been (1) to give the student a particular knowledge of mathematics, (2) to increase the student's appreciation of mathematics, and (3) to discipline the student's mind. While the third goal is largely disregarded today, the first two goals seem to be widely supported (cf. Educational Services Incorporated, 1963). A more complete statement of goals for mathematics instruction is found in a paper authored by R. C. Buck. Those goals are:

1. To provide understanding of the interaction between mathematics and reality.
2. To convey the fact that mathematics, like everything else, is built upon intuitive understandings and agreed conventions, and that these are not eternally fixed.
3. To demonstrate that mathematics is a human activity, that its history is marked by inventions, discoveries, guesses, both good and bad, and

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that the frontier of its growth is covered by interesting unanswered questions.

4. To contrast "argument by authority" and "argument by evidence and proof"; to explain the difference between "not proved" and "disproved," and between a constructive proof and a nonconstructive proof.

5. To demonstrate that the question "Why?" is important to ask and that in mathematics an answer is not always supplied by merely giving a detailed proof.

6. To show that complex things are sometimes simple and simple things are sometimes complex; and that, in mathematics as well as in other fields, it pays to subject a familiar thing to detailed study, and to study something which seems hopelessly intricate. (Buck, 1965, pp. 949 ff.)

The first goal implies that a modern elementary mathematics program should be based in reality and that the child should have considerable experience in translating reality into mathematics and conversely. Thus, such a program will need to teach the child to observe and classify the world around him, to represent the observed properties of objects and sets as mathematics, and to make a mathematical statement and interpret it in reality. For the arithmetic sequence this means that the sequence should begin with a unit on the classification of objects and sets with respect to properties which are measured. Since most real world measures are of continuous properties such as length, weight, area, and volume, the initial unit should deal with some of these



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properties. In subsequent units this goal implies that along with the mathematics a variety of real world models which that bit of mathematics represents should be studied. Thus the arithmetic sequence should be a constructive one; that is, since it must begin within the child's prior experience of the real world and since his conception of that world is simple and only partially complete, the first arithmetic studied should be as simple as possible and that knowledge should be used to construct larger sets and their arithmetics.

Together goals 1, 3, and 5 imply that, while the eventual outcome of arithmetic might be an axiomatic study of number systems and modern algebra, the initial emphasis should be on the intuitive discovery of those number systems, that the child should be the inventor and not merely a spectator, that all definitions should, if possible, be formulated by the child or be well connected to his experience, and that the arithmetic topics should be arranged so as to make this possible. Thus goals 1, 3, and 5 strongly reinforce the conclusions reached from goal 1 and may seem to more strongly indicate the pedagogic approach to be used. However, the author believes that these goals imply a great deal about instructional sequencing. For example, before being presented a unit dealing with small positive integers and mathematical sentences of the forms $a = b$, $a \neq b$, $a < b$, and $a \pm x = b$, the child should have extensive experience comparing objects with respect to a given property, comparing sets on the property of numerousness, and equalizing two objects or sets by "putting with" the smaller and "taking away from" the larger (Romberg, Fletcher and Scott, 1968). As a second example, consider the algorithms for addition and subtraction of positive integers in compact

notation. Instead of beginning with two numbers in compact notation and dictating a set of rules for addition to the child, these goals would suggest that the child should begin by partitioning sets into subsets of prescribed size. After he had learned how to symbolize the results of these partitionings, he would next discover how to add and subtract numbers expressed in expanded notation using sets to validate his results, and finally he would develop rules for adding and subtracting pairs of compact notation numbers using both his expanded notation techniques and his experiences with sets to validate the results of his actions.

Goals 4 and 5 together imply that in a modern elementary mathematics program the child should be asked to question any result he discovers or is given and be encouraged to verify or validate that result. However, since it seems inappropriate to teach primary grade children how to write logically reasoned proofs and since goal 5 states that a detailed proof is not always the answer, in the early school years a modern elementary mathematics program should rely upon empirical validation. In a program dealing simultaneously with reality and mathematics, this seems to be a highly appropriate means of proving results since only recently have the physical sciences begun to use any other means of establishing their results; the biological sciences depend almost exclusively upon this technique, and the social sciences have not progressed far beyond the observation of phenomena. Of course, in the upper elementary grades it seems possible that the child could be taught to construct proofs; however, there is little evidence to substantiate this. The research conducted by King (1970) does indicate some possibilities with college-capable children and highly competent teachers. The research by Suppes and Hill (1962) trained high

achieving children in symbolic logic with good results, but there is little evidence that this knowledge transferred to the construction of mathematical proofs. Thus it seems that the sequence of topics should be arranged so as to make empirical validation possible; this conclusion implies a program rich in models and one in which some of the models are introduced before the mathematics and are repeatedly used both before and after the mathematical concept is established.

But how do these goals and the implications drawn from them correspond to the ideas currently propounded by developmental psychology? It seems to the author that they are compatible with the developmental ideas of both Jean Piaget and Jerome Bruner. Piaget describes the global development of children when he postulates four stages: the sensory-motor stage (0 - 2 years), the preoperational representation stage (2 - 7 years), the concrete operational stage (7 - 11 years), and the formal operational stage (11 - 15 years) (Flavell, 1963).

Bruner theorizes that there are three ways in which human beings represent experience: enactive, iconic, and symbolic (Bruner, 1966). Using this approach each learning experience should give the child an opportunity to interact with reality, to deal with an "icon" or picture of reality, and finally to represent his experience symbolically. The key to constructing such experience is in knowing what is and what is not real to the child. Since the Piagetian stages seem to roughly describe the child's changing conception of reality, the two theories seem to complement one another; one describes the sequence of learning activities for a concept while the other describes the level of the activities. Also in combination it is seen that the conclusions reached using the Buck goals are the same as those reached here.

The investigations of Piaget are helpful in another way. It was concluded earlier in this paper that continuous properties such as length, weight, area, and volume should be studied and, eventually, ways in which measures can be assigned to them should be investigated. With regard to this matter the conservation studies of Piaget lead the author to the conclusion that the sequence in which these properties should appear is precisely that given above and that the introduction of area and volume should not occur until about age eight or nine (Piaget, Inhelder, and Szeminska, 1960).

In summary, the goals for instruction which have been chosen and the information obtainable from developmental psychology indicates the following with respect to the sequencing of the arithmetic topics:

1. The mathematics studied initially should be simple and gradually become more complex; i.e., instead of beginning with the rational numbers and their relations and operations and then successively studying that system's subsystems, the study should begin with small counting numbers and progressively grow to include the arithmetic of the rational numbers.
2. Each time a new mathematical concept is to be acquired the child should have the opportunity to pass from the enactive through the iconic to the symbolic representation stage thus providing a link between reality and mathematics.
3. Conversely, to model mathematics in reality, aid in the development of mathematical intuition,

and to assert the need for proof, each concept unit should require the student to validate his results.

4. In order to make 2 and 3 possible a wide variety of physical models should be introduced and studied.

5. The conclusions of Piaget with respect to the concepts of length, weight, area, and volume should be heeded.

Viable Instructional Sequences

In this section three viable instructional sequences for the arithmetic of the rational numbers are proposed, based upon the preceding discussion. The general procedure to be used will be as follows:

1. When the description of a unit has been completed, all units which might follow it will be discussed and certain of them will be eliminated. If more than one unit remains, one will be chosen and the sequence outline will continue to a conclusion. Then the paper will return to that juncture, another unit will be selected, and the procedure will be repeated.

2. Each unit description will be in two parts: (a) a fuller discussion of the content included; and (b) the titles of the topics included in the unit.

Unit One

In the discussion of goal 1 (page 17) it was concluded that the arithmetic sequence should begin with the classification of objects and

sets with respect to properties to which number measures can be assigned. In addition it was concluded that the initial unit should include a measurable continuous property. Recalling the results of Piaget's investigations it seems that the continuous property initially introduced should be length. Thus Unit One will focus primarily on the property of length and on the property of numerousness of sets. Since the simplest arithmetic operations are addition and subtraction, within this unit the child will learn to compare and equalize objects with respect to their length, and sets with respect to their numerousness. Since it has also been concluded that the study should always proceed from concrete experiences to symbolic representation, the initial unit will begin with concrete experiences, proceed to pictorial representation of those experiences, and conclude with symbolic representation. Finally, as Romberg, Fletcher, and Scott (1968) argue in their paper, length will be introduced before numerousness because comparison and equalization of this property is more natural to the child; large numerals will not be included so as not to concern the child with place value until he is more familiar with the operations and relations. Thus the topic outline for Unit One is:

| Unit One Topic | |
|---|---|
| 1 | Identifying properties of objects |
| 2 | Classifying and describing objects |
| 3 | Identifying length as a property of objects and comparing objects on length |
| 4 | Equalizing objects on length |
| 5 | Ordering objects on length |
| 6 | Representing length physically |

7 Representing length pictorially

8 Classifying and describing sets

9 Identifying numerousness as a property of sets and comparing sets on numerousness

10 Equalizing sets on numerousness

11 Ordering sets on numerousness

12 Representing numerousness physically

13 Representing numerousness pictorially

14 Tallying units of length

15 Counting from 0 to 10

16 Recognizing the numerals 0-10

17 Comparing two objects or sets by counting

18 Identifying weight as a property of objects, comparing, equalizing, and ordering objects on weight, and representing weight physically and pictorially.

19 Comparing two objects or sets using the symbols 0-10, =, and ≠

20 Writing the numerals 0-10

21 Writing comparison sentences using the numerals 0-10 and the symbols = and ≠

22 Comparing numerals representing objects and sets

23 Writing and validating the equalization sentence $a = b + x$

24 Writing and validating the equalization sentence $a = b - x$

25 Constructing, validating and using an equalization table

26 Writing and validating the number sentences $a + b = x$ and $a - b = x$

Unit Two

At the conclusion of Unit One the child will be able to determine whether two small numbers are equal or unequal and to express that relationship symbolically. He will be able to complete any one of the sentences $a \pm x = b$ and $a \pm b = x$ as long as the sum or the minuend does not exceed ten. Thus the logical topics for inclusion in Unit Two are those which lead to and include the algorithms for addition and subtraction of larger numerals. As was indicated earlier, in order to enable the child to generate and understand these algorithms it will be necessary for him to first add the same numbers expressed in expanded notation. In order that expanded or compact numerals have meaning for the child he will have extensive experience partitioning sets with respect to a fixed base b. Initially b cannot exceed ten since the child will not have had experience with any larger numbers. The order will be included by ordering partitioned sets, then expanded numerals, and finally compact numerals. Grouping experiences also provide a valuable type of incidental learning. Since this unit will result in the student writing sentences of the type $35 = 7(5)$ and $41 = 6(6) + 5$ some valuable early experience can be gained about the multiplication and division of positive integers. Additionally, grouping using one base, regrouping using a second base, and then symbolizing the relationship between the two groupings emphasizes the invariance of the numerousness of the set contrasted with the changing conventions which can be used to represent that number. These kinds of activities will be

used to teach relationships between various standard measures; e.g., ounces, pints, quarts, and gallons; seconds, minutes, hours, days, weeks, months and years; ounces, pounds, and tons; inches, feet, yards and miles; centimeters, meters, and kilometers. Finally, this unit must complete the child's knowledge of the basic facts before proceeding to the algorithms. Thus the topic outline is as follows:

| Unit Two Topic | |
|---|---|
| 1 | Grouping by base b ($b \leq 10$) and writing a phrase representing that grouping |
| 2 | Recognizing and ordering the numerals 0-20 and using those numerals to enumerate sets of objects and to describe the length and weight of objects |
| 3 | Writing and validating order, equalization, addition and subtraction sentences using the numerals 0-20 |
| 4 | Grouping by base b and by base c ($b,c \leq 20$) and writing the mathematical sentences $ab + d = ec + f$ |
| 5 | Writing numerals in compact notation and the sentence $10x + y = xy$ and ordering larger numerals |
| 6 | Writing and validating addition and subtraction sentences having 2-digit expanded numerals as summands, or as subtrahends and minuends |
| 7 | Writing and validating addition and subtraction sentences having 2-digit compact numerals as summands or as subtrahends and minuends |
| 8 | Writing and validating mathematical sentences of the form $a \pm x = b$, $a < b$, $a \pm c < b \pm c$ using 2-digit numerals |

9 Grouping sets using a fixed base and writing expanded and compact numerals having three or more digits.

10 Writing and validating the mathematical sentences $a \pm b = x$, $a \pm x = b$, $a < b$, and $a \pm c < b \pm c$ using numerals having three or more digits

Unit Three

Traditionally the topics which would be considered immediately after completing the study of the addition and subtraction algorithms for the counting numbers would be (a) definition of multiplication and division for the counting numbers and (b) development of the algorithms for these operations. There are two other units which might be developed next: (1) a unit on the integers which would define these numbers and would study the equality, order relations, and operations of addition and subtraction; or (2) a unit on non-negative rational numbers which would define these numbers and would study the equality, order relations, and operations of addition and subtraction.

As a matter of fact the unit on the integers is chosen over the alternatives for the following reasons:

1. The counting numbers under addition are incomplete in many ways. A natural next step, algebraically, is to define an opposite for each of these numbers, thereby producing the integers.

2. There is some evidence that a young child does not readily attend to two attributes simultaneously (Piaget and Inhelder, 1956; Howard and Templeton, 1966). Thus, any concepts which will require the child to do so should be delayed as long as possible. Aside from the repeated additions model, all of the physical and pictorial models for multiplication of

counting numbers are two dimensional; it is the two dimensional models of multiplication which are especially usable when the number system is extended to the rational numbers or the real numbers. Thus if a child is to interpret multiplication so that his idea can be extended, it is necessary that he be able to attend to two attributes simultaneously. For every number system being considered in this paper, addition can be accurately modeled as a one-dimensional process.

3. While it is true that the child was introduced to the concept of place value in Unit Two, exponentiation was not discussed at all. Thus the non-negative rational numbers would need to be introduced as common fractions. The only available model is the subdivision of the intervals on the number line and congruent subdivision of "pies." Additionally, since the definitions of equality, order, and addition of rational numbers depend heavily upon multiplication of integers, definition of these relations and this operation would seem to be impossible at this point.

When there is the desire to begin to study each concept in the child's world, to move to a pictorial representation of that world, and finally to arrive at the mathematical abstraction--and if the integers are defined as sets of vectors which are of the same length and direction on a given line--the following topic outline emerges:

Unit Three
Topic

1 Physical models for vectors and for the integers

2 Identification of straight line segments, of the properties of direction and opposite direction, of points on a line segment, and of movement from a point

3 Pictorial representation of movement in a direction and in the opposite direction and construction of the counting number ray

4 Identification of vectors, equivalent vectors, and opposite vectors

5 Definition of the negative integers, construction of the integer number line, and ordering the integers

6 Writing and validating addition, subtraction, equalization, and order sentences using integers

Before concluding the discussion of this unit it should be noted that a vector is a two attribute concept. Beginning with Unit One the child has had extensive experience with the property of length, therefore the addition of the property of direction should not prove too burdensome. Also, two dimensions are not necessary to represent vectors.

At the conclusion of this unit the child will have studied the integers, the order properties and the additive structure. Perhaps the time has arrived to introduce multiplication and division.

Unit Four

Actually at this point in the sequencing of the arithmetic topics the only choice which seems feasible is a unit on the definition of multiplication and division of integers and on the algorithms associated with these operations. The other alternative, a unit on the addition and subtraction of rational numbers, is eliminated for the same reasons given in the discussion of Unit Three. Thus the topic outline for Unit Four is:

| Unit Four Topic | |
|---|---|
| 1 | Two models for multiplying and dividing pairs of positive integers--the repeated additions and subtractions model and the grouping and partitioning of sets model |
| 2 | The measurement of area and volume. A third model for the multiplication and division of positive integers |
| 3 | Some multiplicative properties--the distributive property, the associative property, the commutative property, and the properties of 0 and 1 |
| 4 | Writing and validating mathematical sentences of the form $ab = c$ and $a \div b = c$ for appropriately small integers |
| 5 | Exponentiation, the identification of prime numbers, and the multiplicative order properties |
| 6 | The multiplication algorithm for pairs of positive integers |
| 7 | The division algorithm for pairs of positive integers |
| 8 | A model for the multiplication and division of any pair of integers - the line model |
| 9 | The multiplication and division algorithm for pairs of integers |
| 10 | Prime factorization of integers, the calculation of the LCM and GCD, and divisibility tests |
| 11 | More on multiplication and division algorithms |
| 12 | The integers - a summary |

The last topic in this unit is intended to summarize and review all of the previous units including this one. This summary is appropriate since

this unit concludes the study of the integers and since the child can demonstrate that any integer property holds for given specific integers. Thus the last unit will concern itself with defining the set of rational numbers and its relations and operations.

Unit Five

The mathematician interchangeably uses two different representations each of which he calls the rational numbers. One representation is the set of common fractions. The other is the set of repeating decimal fractions. The average adult uses both of these representations and sometimes moves from one to the other but not with much facility or understanding; as a matter of fact he rarely, if ever, uses the entire set of repeating decimals but restricts his attention to those which terminate. Since it seems important to study both representations and the means of converting from one representation to the other and since the rational numbers are a complex mathematical structure, great care must be exercised in the instructional sequencing. Three approaches seem viable:

Alternative A: Introduce the rational numbers as the common fractions and subsequently introduce and study the decimal fractions.

Alternative B: Simultaneously study common fractions and decimal fractions.

Alternative C: Introduce the rational numbers as decimal fractions and subsequently introduce and study the common fractions.

Alternatives A and C have the same advantages: efficiency, a basis in psychology and a basis in mathematics. The most efficient and fastest

means of completing the study of the arithmetic of the rational numbers is to sequence the content as in Alternative A or C. Since the elementary mathematics program being proposed is an ambitious one, choosing one of these alternatives might be appropriate since adjustments in the length may be necessary--a contingency which it is hoped would not develop. Secondly, there is a psychological advantage in Alternatives A or C in that the child would not be confronted with as many new concepts simultaneously as he would be should both the common fractions and the decimal fractions be introduced almost simultaneously. And the problems related to dealing with the correspondence between the two representations would not occur until somewhat later in the program, after the child has become well acquainted with at least one of the means available for describing the rational numbers. Finally, since the two systems are alike except in form it is appealing mathematically to introduce one structure and study its other manifestations later--a sort of progression from example to general case which so often occurs in mathematics.

The advantages of Alternate B could be called the disadvantage of Alternate A and of Alternate C. First, an accurate description of the real numbers which a child might understand would be that they are infinite decimal fractions. Thus the child should have familiarity with decimal representations of rational numbers if he is to obtain intuitive understandings of the real numbers. Secondly, the definition of multiplication of rational numbers is very easy using common fractions, but the definition of addition of rational numbers is very difficult to justify or motivate when this representation is used. Thus a definite advantage of Alternate B would be the ability to go from one representation to another at will in

order to obtain easily motivated and well-founded definitions. Finally, since decimal representations of rational numbers are more easily used in estimation and precision of measurement, criterion 1 (page 6) would argue that decimal representations should be included as early as possible.

The three topic outlines are:

Unit Five - Alternate A
Topic

1 Physical and pictorial models for common fractions

2 Common fractions, equivalent common fractions, and comparion of common fractions

3 Writing and validating equalization, addition, subtraction, and order sentences using common fractions

4 Writing and validating multiplication, division, and order sentences using common fractions

5 Approximating common fractions with finite sums of base ten common fractions

6 Quotients involving decimal fractions

7 Decimal representation of the rational numbers and the correspondence to the common fractions

8 Writing and validating addition, subtraction, and order sentences using decimal representations

9 Writing and validating multiplication, division, and order sentences using decimal representations

10 The rational numbers - a summary

Unit Five - Alternate B
Topic

1 Physical and pictorial models for common fractions

2 Measurement models for decimal representation of the rational numbers

3 Fractions, equivalent fractions and comparison of fractions

4 Writing and validating equalization, addition, subtraction and order sentences using rational numbers

5 Writing and validating multiplication, division, and order sentences using rational numbers

6 The rational numbers - a summary

Unit Five - Alternate C

Topics

1 Measurement models for decimal representation of the rational numbers

2 Quotients involving decimal fractions

3 Writing and validating addition, subtraction and order sentences using decimal representations

4 Writing and validating multiplication, division and order sentences using decimal representations

5 Representing decimal fractions using common fractions

6 Common fractions, equivalent fractions and comparison of fractions

7 Writing and validating equalization, addition, subtraction and order sentences using fractions

8 Writing and validating multiplication, division, and order sentences using fractions

9 The rational numbers - a summary

V

## CONCLUSION

This paper has shown that the study of arithmetic must be included in a modern elementary mathematics program. It has established criteria which help to select the arithmetic to be included in such a program and, using those criteria, has proposed the study of the arithmetic of the rational numbers as a worthwhile and feasible goal. Finally, the sequence of the topics has been considered and three viable instructional sequences have been suggested.

In conclusion the author extends the hope that this paper will be useful in the development and refinement of modern elementary mathematics programs.

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